Week 5
Game Theory

Lectures from: Robert Gibbons, Game Theory for Applied Economists, Princeton University Press: New Jersey, 1992

1. Normal-form Games

In a normal-form game, each player chooses a strategy (without knowledge of the strategy chosen by the other players) from his 'strategy set' and the pay-off to each player depends on the strategies chosen by *all* the players.
There are N players, i=1…N. The set of feasible strategies available to player i is represented by his *strategy set* S_i (i=1…N); $s_i \in S_i$ is the strategy chosen by the i^th player from his set of feasible strategies; $u_i(s_1,...s_N)$ is the pay-off to the ith player given his own choice of strategy *and the choices made by the other players*. The Game is represented by: $G = \{S_1...S_N; u_1...u_N\}$

1.1 Strictly Dominated Strategies

Game 1.1: The Prisoners' Dilemma

Prisoner 1		*Prisoner 2*	
		Confess	Deny
	Confess	3,3	0,6
	Deny	6,0	1,1

Prisoners 1 and 2 have to decide his best strategy: confess or deny
For P1: if P2 denies, his best strategy is to confess; if P2 confesses, his best strategy is to confess
For P2: if P1 denies, his best strategy is to confess; if P1 confesses, his best strategy is to confess

In a NFR game, ***G***, strategy $s_i' \in S_i$ is **strictly dominated** by another strategy $s_i'' \in S_i$, if for every feasible strategy $s_j \in S_j, j = 1...N, j \neq i$ chosen by the other players:

$$u_i(s_1..s_i''..s_N) > u_i(s_1..s_i'..s_N)$$

Rational players do not play strictly dominated strategies: player i would not play s_i' in the presence of a dominant strategy, s_i''.

In the Prisoners' Dilemma, the strategy 'Confess' is the dominant strategy for both players. So neither player would choose 'Deny'. So the rational choice for both prisoners is to confess. **But this not the Pareto-optimal outcome which would be for both to deny.** The PO outcome cannot be attained because each cannot be sure what the other will choose: **there is an absence of trust!**

Interpret:

1. confess as ‘build a nuclear bomb’; deny as ‘do not build a nuclear bomb’
2. confess as ‘produce more than your quota’; deny as ‘do not produce more than your quota’.

1.2 Iterated Elimination of Strictly Dominated Strategies (IESDS)

Game: 1.2

		Player 2		
		Left	Middle	Right
Player 1	Up	1,0	1,2	0,1
	Down	0,3	0,1	2,0

Show for Player 1 that neither ‘up’ nor ‘down’ is strictly dominated.
Show for player 2 that ‘right’ is strictly dominated by ‘middle’
Show that ‘right’ can be eliminated from Player 2’s strategy set.

Game 1.2A: Game 1.2 After Elimination of Dominated Strategy

		Player 2	
		Left	Middle
Player 1	Up	1,0	1,2
	Down	0,3	0,1

Show that for Player 1, ‘down’ is strictly dominated by ‘up’.

Game 1.2B: Game 1.2A After Elimination of Dominated Strategy

		Player 2	
Player 1		Left	Middle
	Up	1,0	1,2

Show that for Player 1, ‘left’ is strictly dominated by ‘middle’.
So ‘up, middle’ is the chosen strategy.
It is arrived at by a process called the ***iterated elimination of strictly dominated strategies***

Steps:
1. Player 1 knows that Player 2 is rational, and will never play ‘right’
2. Player 2 knows that Player 1 is rational, and will never play ‘up’

General assumption for this process to work is **common knowledge that all players are rational.**

Show that in Game 1.3, all the strategies survive IESDS: there are no strictly dominated strategies:

Game 1.3

		Player 2		
		Left	Centre	Right
Player 1	Top	0,4	4,0	5,3
	Middle	4,0	0,4	5,3
	Bottom	3,5	3,5	6,6

A drawback of IEDS is that more than one (including *all* strategies) strategy may survive.

1.3 Nash Equilibrium

Suppose that GT makes a unique prediction $\hat{s}_i$ about the strategy chosen by player I (I=1…N); If this prediction is to be true, then it must be that $\hat{s}_i = s_i^* \ \forall i$ where s_i^* is player I's 'best response' strategy, *given the predicted strategies* $\hat{s}_j, j = 1...N, j \neq i$ *of all the other players*. Such a prediction is *strategically stable* or *self-enforcing* because no player will want to depart from the predicted strategy and it is termed a ***Nash equilibrium***.

The strategies $s_1^*, s_2^*...s_N^*$ represent a Nash equilibrium if, *for all I=1…N*, s_i^* is player i's 'best-response' to the strategies $s_j^*, j = 1...N, j \neq i$ specified for the other players. That is:

$$u_i(s_1^*..s_{i-1}^*, s_i^*, s_{i+1}^*...s_N^*) \geq u_i(s_1^*..s_{i-1}^*, s_i, s_{i+1}^*...s_N^*)$$

for all $s_i \in S_i$ and for all $i = 1...N$

A proposed solution $\hat{s}_1, \hat{s}_2...\hat{s}_N$ which is *not* a Nash equilibrium cannot be a solution to the game since at least one agent will have an incentive to depart from the solution and the solution will be falsified by the games' outcome: since $\hat{s}_1, \hat{s}_2...\hat{s}_N$ is *not* a Nash equilibrium there will, by definition, exist a strategy $\tilde{s}_i$ for some agent I, such that:

$$u_i(\hat{s}_1..\hat{s}_{i-1}, \hat{s}_i, \hat{s}_{i+1}...\hat{s}_N) < u_i(\hat{s}_1..\hat{s}_{i-1}, \tilde{s}_i, \hat{s}_{i+1}...\hat{s}_N)$$

A solution which is a Nash equilibrium is less demanding than a solution which is a dominant strategy.
In a dominant strategy, A's strategy must be best for *all* choices made by B and B's strategy must be best for *all* choices made by A. In a Nash equilibrium, A's strategy must be best for *the best* choice made by B and B's strategy must be best for *the best* choice made by A.

If the iterated elimination of strictly dominated strategies eliminates all but one strategy vector $\mathbf{s}^* = s_1^*, s_2^*...s_N^*$, then $\mathbf{s}^*$ is a Nash-equilibrium.
Show that the iterated elimination of strictly dominated strategies from Games 1.1 and 1.2 will result in a Nash equilibrium.
Conversely, if $\mathbf{s}^* = s_1^*, s_2^*...s_N^*$ is a Nash equilibrium then it will survive an IESDS.

But there can be strategies that survive IESDS but which are not Nash equilibrium. Show that in game 1.4, a Nash equilibrium ('Top-Left') exists even though all strategies survive IESDS: there is no strictly dominated strategy.

Game 1.4

		Player B	
Player A		Left	Right
	Top	2,1	0,0
	Bottom	0,0	1,2

Show that in Game 1.3 there is a Nash equilibrium (B,R), even though all strategies survive an IESDS.

1.4 Applications

A. Cournot Model of Oligopoly

In a Cournot oligopoly model, the players are the N firms (producing a homogenous product) and the strategies are their choice of outputs, $y_i,\ i = 1...N$ which they choose from their strategy sets: [0,α]. The payoffs to the firms are:

$$\pi_i(y_1..y_i..y_N) = p(Y)y_i - C_i(y_i) \tag{1}$$

where: $C_i(y_i)$ is the cost function of firm i and $Y = \sum_i y_i$.

The outputs $y_1^*...y_N^*$ represent a Nash equilibrium if y_i^* maximises π_i, given that the other firms have chosen $y_j^*, j = 1...N, j \neq i$. So the first-order conditions for a Nash equilibrium are:

$$p(Y^*) + y_i^* p'(Y^*) - C_i'(y_i^*) = 0 \tag{2}$$

which can be rewritten as:

$$p(Y^*)\left[1 + \frac{s_i^*}{\varepsilon}\right] = C_i'(y_i^*) \tag{3}$$

where: ε is the elasticity of demand with respect to price and $s_i^* = \frac{y_i^*}{Y^*}$ is share of firm i in total output. Under monopoly, $s_i^* = 1$ and: $p(Y^*)\left[1 + \frac{1}{\varepsilon}\right] = C_i'(y_i^*)$ while under competition, $s_i^* \to 0$ and $p(Y^*) \to C_i'(y_i^*)$. So, in that sense, oligopoly is "in-between" monopoly and competition.

If all the firms have the *same* cost function, $C(y_i^*)$ then $s_i^* = 1/N$ and equation (3) can also be written as:

$$p\left[1 + \frac{1}{N\varepsilon}\right] = C'(y_i^*) \tag{4}$$

with N=1 representing the monopoly outcome and N→∝ representing the competitive outcome.

Alternatively, equation (4) may be expressed as:

$$\mu = \frac{p(Y^*) - C'(y_i^*)}{p(Y^*)} = -\frac{1}{N}\left[\frac{Y^*}{p(Y^*)}\frac{dp(Y^*)}{dY^*}\right] = -\frac{1}{N\varepsilon} \tag{5}$$

and μ is the Lerner index of 'market power', expressed as the percentage mark up of price over marginal cost.

B. Betrand Model of Oligopoly: Homogenous product

There are two firms (1 and 2) producing a homogenous product, which they price at p_1 and p_2, respectively, and they face marginal costs

$c_1 = C_1'$ and $c_2 = C_2'$, respectively: $c_2 > c_1$. *D(p)* is the aggregate demand curve. The demand curve facing firm 1 is:

$$d_1(p_1, p_2) = \begin{cases} D(p_1), \text{ if } p_1 < p_2 \\ D(p_1)/2, \text{ if } p_1 = p_2 \\ 0, \text{ if } p_1 > p_2 \end{cases} \tag{6}$$

So, firm 1 believes it can capture the entire market by pricing lower than firm 2; firm 2 believes likewise.

What is the Nash equilibrium? Firm 1 *will not* choose $p_1 > c_2$: if it did, firm 2 would choose p_2 such that: $c_2 < p_2 < p_1$ driving 1 out of the market. Firm 1 *will not* choose $p_1 < c_2$ because that would be leaving 'money on the table': at such a price, firm 2 produces zero, but firm 1 could have increased its profits by raising price. So the Nash equilibrium is for firm 1 to set $p_1 = c_2$. At this price, firm 1 produces $D(c_2)$ output, firm 2 sets price $p_2 \geq c_2$ and produces nothing. If both firms had the same marginal cost: $c_1 = c_2 = c$, $p_1 = p_2 = p$ and each firm produces: *D(p)/2*.

C. Betrand Model of Oligopoly: Differentiated product

There are N firms, *i=1…N*, producing a differentiated product. The price charged by firm *i* is p_i and the demand for the product of firm i is $y_i(p_1 \ldots p_i \ldots p_N)$. The payoff to firm *i* from choosing p_i is its profits:

$$\pi_i(\mathbf{p}) = p_i \times y_i(\mathbf{p}) - C_i(y_i(\mathbf{p})) \tag{7}$$

where: $\mathbf{p} = (p_1 .. p_i .. p_N)$.

Then the price vector: $\mathbf{p}^* = \{p_i^*\}$ is a Nash-equilibrium, if for $\forall$i, p_i^* solves:

$$Max\ \pi_i(p_1^* .., p_{i-1}^*, p_i, p_{i+1}^* .. p_N^*) \tag{8}$$

From (7), the first-order conditions for this are:

$$\frac{\partial \pi_i(p_1^* .. p_i .. p_N^*)}{\partial p_i} = y_i(p_1^* .. p_i .. p_N^*) + p_i \frac{\partial y_i(p_1^* .. p_i .. p_N^*)}{\partial p_i} - \frac{\partial C_i}{\partial y_i} \times \frac{\partial y_i(p_1^* .. p_i .. p_N^*)}{\partial p_i} = 0 \tag{9}$$

and this can be rewritten as:

$$\begin{aligned} & y_i(p_1^* .. p_i .. p_N^*)\left[1 + \varepsilon_i\right] = c_i \frac{y_i(p_1^* .. p_i .. p_N^*)}{p_i} \varepsilon_i \\ & \Rightarrow p_i^* = c_i \frac{\varepsilon_i}{1 + \varepsilon_i} \end{aligned} \tag{10}$$

where: $c_i = \partial C_i / \partial y_i$ is the marginal cost of firm *i* and ε_i is the price-elasticity of demand for firm *i*.

C. The Tragedy of the Commons

There are N farmers in a village who graze their cows on the village green. This is owned in common by all the villagers. The number of goats owned by the ith farmer is g_i and $G = \sum_i g_i$ is the total number of goats grazing on the green. The price of a goat is *c* and *v(G)* is the value of the milk from a goat when there are *G* goats grazing.

The maximum number of goats that can be grazed on the green is $\bar{G}$: *v(G)=0* if G= $\bar{G}$, while *v(G)>0* if G< $\bar{G}$
The 'strategy' for each farmer is to choose his g_i from his 'strategy set': *[0,∝]* . The payoff to the farmer
from g_i goats depends upon his choice, as well as upon the choices made by others:

$$\pi_i = g_i v(g_1 + .. + g_{i-1}^* + g_i + g_{i+1}^* + ... + g_N) - cg_i \tag{11}$$

If $g_1^* ... g_N^*$ is to be a Nash-equilibrium, then, for each farmer *i, i=1…N,* g_i^* should maximise π_i , given that the other farmers choose $g_j^*, j = 1...N, j \neq i$. The first-order conditions for maximising π_i wrt g_i are:

$$v(g_i + \mathbf{g}_\mathbf{j}^*) + g_i v'(g_i + \mathbf{g}_\mathbf{j}^*) - c = 0 \tag{12}$$

where: $\mathbf{g}_\mathbf{j}^* = \sum_{\substack{j=1 \\ j \neq i}}^{N} g_j^*$ and, for a Nash equilibrium, g_i^* solves (1). Consequently, the conditions for a Nash equilibrium are:

$$v(G^*) + g_i^* v'(G^*) - c = 0,\ i = 1...N \tag{13}$$

where: $G^* = \sum_{i=1}^{N} g_i^*,\ v'(G^*) = \frac{\partial v(G^*)}{\partial G^*}$

Interpretation: There are G^* goats being grazed, so payoff per goat is *v(G*)*. A farmer is contemplating adding a goat. This goat will give a payoff of $v(G^*)$ but it will reduce the payoff from his existing goats by the reduction in the payoff-per-goat (after another goat has been added), $v'(G^*)$, times the number of goats he owns, g_i^*. This is his ***marginal private benefit*** from adding another goat. He compares this marginal private benefit ($v(G^*) + g_i^* v'(G^*)$) to the cost of a goat, *c*, and decides accordingly. Note that when $g_i^* = 1$ (he owns only one goat), $v(G^*) + g_i^* v'(G^*)$ is the new **average payoff** from a goat.

Summing over the farmers' first-order conditions and dividing by N, yields:

$$v(G^*) + (G^* / N) v'(G^*) - c = 0 \tag{14}$$

and solving (3) yields the Nash equilibrium (total) number of goats, G^*

In contrast, the social optimum number of goats, G^{**} is given by maximising the net revenue to the village from the goats:

$$Gv(G) - Gc \tag{15}$$

and this yields as first-order conditions:

$$v(G^{**}) + G^{**} v'(G^{**}) - c = 0 \tag{16}$$

Interpretation: There are G^{**} goats being grazed, so payoff per goat is *v(G**)*. The village is contemplating adding a goat. This goat will give a payoff of $v(G^{**})$ but it will reduce the payoff from the existing goats in the village by the reduction in the payoff-per-goat (after another goat has been added), $v'(G^{**})$, times the number of goats in the village, G^{**}. This is the ***marginal social benefit*** from adding another goat. The village compares this marginal private benefit ($v(G^{**})+G^{**}v'(G^{**})$) to the cost of a goat, *c*, and decides accordingly.

Comparing (3) with (5) shows that $G^{*}>G^{**}$. The common resource is over utilised because each villager considers the effect of his action (of grazing another goat) on only his own welfare and neglects the effect of his action upon the other villagers (compare (1) with (5)).

1.5 Mixed Strategies

Nash equilibrium has certain problems.

First, an equilibrium, even if it exists, *may not be unique*. This is the problem of multiple Nash equilibrium.

Show that in Game 1.5 (identical to Game 1.4), *both* (opera, opera) and (football, football) are Nash equilibrium.

Game 1.5: Battle of the Sexes

		Player B	
Player A		Opera	Football
	Opera	2,1	0,0
	Football	0,0	1,2

Second, a Nash equilibrium may not exist.

Show that in Game 1.6, there is no Nash equilibrium.

Game 1.6

		Player B	
Player A		Left	Right
	Top	0,0	0,-1
	Bottom	1,0	-1,3

If A plays top, B plays left; but if B plays left, A plays bottom.
If A plays bottom, B plays right; but if B plays right, A plays top.

Each player in Game 1.7, below, has a penny and has to display a face: ‘heads’ or ‘tails’. If the faces match, B wins A’s penny; if the faces are different, A wins B’s penny. Show that in this game of ‘matching pennies’, there is no Nash equilibrium.

Game 1.7: Matching Pennies

		Player B	
Player A		Heads	Tails
	Heads	-1,1	1,-1
	Tails	1,-1	-1,1

Interpretation: A bowler in cricket can either bowl ‘short of a length’ or ‘pitch it up’. A batsmen can either play ‘back’ or ‘forward’. Given the speed at which the bowler is bowling (Bret Lee?), the shot has to be pre-determined.

In a normal-form game G, suppose $S_i = \{s_{i1}...s_{iK}\}$. Then the strategy set S_i for player i consists of the K ***pure strategies***: $s_{ik}, k = 1...K$. Then a ***mixed strategy*** for player i is a probability distribution $p_i = \{p_{i1}...p_{iK}\}, 0 \le p_{ik} \le 1, k = 1...K, \sum_k p_{ik} = 1$, where p_{ik} is the probability that he will play the (pure) strategy s_{ik}.

Can we be sure that a Nash equilibrium exists?
Theorem: Nash, J. (1950), "Equilibrium Points in n-Person Games", Proceedings of the National Academy of Sciences, 36: 48-49.
In any n-player normal-form game $G = \{S_1...S_N; u_1...u_N\}$, there exists at least one Nash equilibrium (possibly involving mixed strategies) if the number of players N is finite and every player's strategy set S_i is finite.

Game 1.8

Player 1		*Player 2*	
		Left	Right
	Top	3,--	0,--
	Middle	0,--	3,--
	Bottom	1,--	1,--

In Game 1.8, 'bottom' is not strictly dominated by either 'top' or 'middle': if 2 plays 'left', then 'top' is better than 'bottom', but if 2 plays right, 'bottom is better than 'top'; if 2 plays 'right', then 'middle' is better than 'bottom', but if 2 plays 'left', 'bottom' is better than 'middle'.
Now consider the **mixed strategy**, *x*: play 'top' with probability=0.5 and play middle with probability=0.5. Then the expected payoff to 1 from *x* is: 3/2 if 2 plays 'left' and 3/2 if 2 plays 'right'. So regardless of what 2 does, the pure strategy 'bottom' is strictly dominated by the mixed strategy *x*, even though it is not strictly dominated by any of the pure strategies 'top' or 'middle'.
Suppose 1 believes that 2 will play 'left' with probability *q* and 'right' with probability *1-q.* Then 1's best response is 'top' if $q \ge 0.5$ and 'middle' if $p \le 0.5$, but *never* 'bottom'.

Game 1.9

Player 1		*Player 2*	
		Left	Right
	Top	3,--	0,--
	Middle	0,--	3,--
	Bottom	2,--	2,--

In Game 1.9, 'bottom' is not the best response to 2's play of the *pure* strategies: 'left' or 'right'. Suppose player 2 follows the *mixed* strategy *y*: play 'left' with probability *q* and 'right' with probability *1-q*. Then the ER from 'top' is 3q and the ER from 'middle' is 3(1-q) and 'bottom' is better strategy for 1 than 'top' or 'middle, provided $1/3 \le q \le 2/3$.

1.6 Nash Equilibrium in the Presence of Mixed Strategies

A Nash equilibrium in *pure* strategies occurs when each player's choice of *pure* strategy is the *best response* to the other players choice of *pure* strategies.
A Nash equilibrium in *mixed* strategies occurs when each player's choice of *mixed* strategy is the *best response* to the other players choice of *mixed* strategies.

Suppose there are two players: 1 and 2. Player 1 and 2's strategy sets are, respectively: $A = \{a_j, j = 1..J\}$ and $B = \{b_k, k = 1..K\}$ where $a_j \in A$ and $b_k \in B$ are the *pure* strategies.
Player 1 believes that 2 will play strategy b_k with probability q_k, $k=1..K$, $\sum q_k = 1, q_k \geq 0$. Then player 1's expected payoff (EP) from playing the *pure* strategy, a_j is:

$$EP(a_j) = \sum_{k=1}^{K} q_k u_1(a_j, b_k) \tag{17}$$

and player 1's EP from playing the *mixed* strategy: $\mathbf{p} = (p_1 ... p_J)$ is:

$$\begin{aligned} EP(\mathbf{p}) = v_1(\mathbf{p}, \mathbf{q}) &= \sum_{j=1}^{J} p_j \left[\sum_{k=1}^{K} q_k u_1(a_j, b_k) \right] \\ &= \sum_{j=1}^{J} \sum_{k=1}^{K} p_j q_k u_1(a_j, b_k) = \sum_{j=1}^{J} p_j EP(a_j) \end{aligned} \tag{18}$$

Note that EP(**p**) is a *weighted* sum of the expected payoffs from the pure strategies (defined in equation (17)).
So for $\mathbf{p} = (p_1 ... p_J)$ to be player 1's best mixed strategy response to player 2's mixed strategy $\mathbf{q} = (q_1 ... q_K)$, $p_j > 0$ only if $EP(a_j) \geq EP(a_j'), \ \forall a_j' \in A$.

Similarly, if Player 2 believes that 1 will play strategy a_j with probability p_j, $j=1..J$, $\sum p_j = 1, p_j \geq 0$, then player 2's expected payoff (EP) from playing the *pure* strategy, b_k is:

$$EP(b_k) = \sum_{j=1}^{J} p_j u_2(a_j, b_k) \tag{19}$$

and player 2's EP from playing the *mixed* strategy: $\mathbf{q} = (q_1 ... q_K)$ is:

$$\begin{aligned} EP(\mathbf{q}) = v_2(\mathbf{p}, \mathbf{q}) &= \sum_{k=1}^{K} q_k \left[\sum_{j=1}^{J} p_j u_2(a_j, b_k) \right] \\ &= \sum_{k=1}^{K} \sum_{j=1}^{J} q_k p_j u_2(a_j, b_k) = \sum_{k=1}^{K} q_k EP(b_k) \end{aligned} \tag{20}$$

The mixed strategies $(\mathbf{p}^*, \mathbf{q}^*)$ represent a Nash equilibrium if:

$$v_1(\mathbf{p}^*, \mathbf{q}^*) \geq v_1(\mathbf{p}, \mathbf{q}^*) \text{ and } v_2(\mathbf{p}^*, \mathbf{q}^*) \geq v_2(\mathbf{p}^*, \mathbf{q}) \tag{21}$$

for every probability distribution **p** in A and every probability distribution **q** in B.

In other words, for a Nash equilibrium in mixed strategies, each player's mixed strategy must be the best response to the other player's mixed strategy.

In the 'matching pennies' Game 1.7, suppose B plays the mixed strategy *y:* 'heads' with probability *q* and tails with probability *1-q*. Given this belief, A's expected payoffs are: -q+(1-q)=1-2q from playing 'heads'; and q-(1-q)=2q-1 from playing 'tails'. If q<0.5, 1-2q>2q-1 and the pure strategy 'tails' is strictly dominated by the pure strategy 'heads'; if q>0.5, 1-2q<2q-1 and the pure strategy 'heads' is strictly dominated by the pure strategy 'tails'.
So A's best pure strategy response to B's mixed strategy (q,1-q) is: 'heads', if q<0.5, 'tails if q>0.5.

Now suppose A plays the mixed strategy *x:* 'heads' with probability *p* and tails with probability *1-p*. For all $0 \le q \le 1$, $p^*(q)$ be the value of *p* such that the mixed strategy $[p^*(q), 1-p^*(q)]$ offers the best response to [q, 1-q].
A's expected payoff from [p, 1-p] is:

$$E(x) = p(1-q) + (1-p)q - pq - (1-p)(1-q) = (2q-1) + p(2-4q) \qquad (22)$$

and since $\partial E(x)/\partial p = (2-4q)$, E(x) is increasing with *p* if *2-4q>0* and decreasing in *p* if *2-4q<0*. So A's best mixed strategy is *p=1*, if *q<0.5* and *p=0*, if *q>0.5*.

So, of all the mixed strategy responses [p(q), 1-p(q)] available to A, A's best response is $p^*(q)=1$ if $q<0.5$ (ie. play the pure strategy, 'heads') and $p^*(q)=0$ if $q>0.5$ (ie. play the pure strategy, 'tails').
Similarly, of all the mixed strategy responses [q(p), 1-q(p)] available to B, B's best response is $q^*(p)=1$ if $p>0.5$ (ie. play the pure strategy, 'heads') and $q^*(p)=0$ if $p<0.5$ (ie. play the pure strategy, 'tails').

The Nash equilibrium is: (H,H) if *q<0.5* and *p>0.5*; (T,H), if *q>0.5* and *p<0.5*; (H,T) if *q<0.5* and *p<0.5*; (T,T) if *q>0.5* and *p>0.5*.

2. Dynamic Games with Complete Information

2.1 Dynamic Games of Complete and Perfect Information

Such games are defined as follows:

1. First, player 1 chooses an action a_1 from his action set A_1.
2. Then, player 2, after observing player 1's choice, chooses an action a_2 from his action set A_2.
3. The payoffs are: $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$

The game is ***dynamic*** because the action of player 2 is taken after knowing the action of player 1.
Information is ***complete*** because the payoff functions are common knowledge.
Information is ***perfect*** because, at each move, the player to move knows the full history of the game.

The problem before player 2, at the second stage of the game, is to choose $a_2 \in A_2$, given a1, so as to maximise: $u_2(a_1, a_2)$. Suppose for every $a_1 \in A_1$, there is a unique optimal $a_2 = R_2(a_1)$. Then R_2 is 2's **reaction function**.

Then player 1's problem at the first stage of the game, is to choose $a_1 \in A_1$, given that $a_2 = R_2(a_1)$ so as to maximise:

$$u_1(a_1, a_2) = u_1(a_1, R_2(a_1))$$

Suppose this optimisation problem has a solution: $[a_1^*, R_2(a_1^*)]$. Then $[a_1^*, R_2(a_1^*)]$ is termed the **backward induction** solution to the dynamic game.

The central issue in dynamic games is the **credibility** of threats. Player 2 "threatens" action a_2 in response to player 1's a_1. But player 1 knows that this is a **credible** threat in that it represents player 2's *optimal response* to 1's action: $a_2 = R_2(a_1)$.

A. Application: Stackelberg Duopoly with Output Leadership

In a Stackelberg duopoly model with output leadership, the players are two firms (producing a homogenous product) and the strategies are their choice of outputs, $y_i,\ i = 1, 2$ which they choose from their strategy sets: $[0,\alpha]$. The payoffs to the firms are:

$$\pi_i(y_1, y_2) = p(Y)y_i - C_i(y_i) \tag{23}$$

where: $C_i(y_I)$ is the cost function of firm I and $Y = \sum_i y_i$.

The difference between the Stackelberg and the Cournot models is that under Cournot the firms move simultaneously while in Stackelberg they move sequentially.

1. Firm 1 chooses an output level y_1.

2. Observing this, firm 2 chooses y_2 so as to maximise its profits:
   $\pi_2(y_1, y_2) = p(Y)y_2 - C_2(y_2)$
3. Firm 1, knowing the reaction function $y_2 = R_2(y_1)$ solves the game by backward induction by choosing y_1 so as to maximise:

$$\pi_1(y_1, R_2(y_1)) = p(Y)y_1 - C_1(y_1) \qquad (24)$$

The first-order conditions for solving this are:

$$\frac{d\pi_1(y_1, R_2(y_1))}{dy_1} = p(Y) + y_1 \frac{dp}{dY}\left[1 + \frac{dR_2(y_1)}{dy_1}\right] - \frac{dC_1(y_1)}{dy_1} = 0 \qquad (25)$$

which yields:

$$p(Y)\left[1 + \frac{s_1}{\varepsilon}\left(1 + R_2'(y_1)\right)\right] = c_1 \qquad (26)$$

B Application: Stackelberg Duopoly Model with Price Leadership

In a Stackelberg duopoly model with price leadership, the players are two firms (producing a differentiated product) and the strategies are their choice of outputs, $p_i,\ i = 1,2$ which they choose from their strategy sets: [0,α]. The payoffs to the firms are:

$$\pi_i(p_1, p_2) = p_i y_i(p_1, p_2) - C_i(y_i) \qquad (27)$$

where: *$C_I(y_I)$* is the cost function of firm i.
The difference between the Stackelberg price leadership and the Betrand models is that under Betrand the firms move simultaneously while in Stackelberg they move sequentially.

4. Firm 1 chooses a price p_1.
5. Observing this, firm 2 chooses p_2 so as to maximise its profits:
   $\pi_2(p_1, p_2) = p_2 y_2(p_1, p_2) - C_2(y_2(p_1, p_2))$
6. Firm 1, knowing the reaction function $p_2 = R_2(p_1)$ solves the game by backward induction by choosing *p_1* so as to maximise:

$$\pi_1(p_1, R_2(p_1)) = p_1 y_1(p_1, R_2(p_1)) - C_1(y_1(p_1, p_2)) \qquad (28)$$

The first-order conditions for solving this are:

$$\begin{aligned} \frac{d\pi_1(p_1, R_2(p_1))}{dp_1} &= p_1\left[\frac{\partial y_1}{\partial p_1} + \frac{\partial y_1}{\partial p_2}\frac{dR_2(p_1)}{dp_1}\right] + y_1(p_1, p_2) \\ &- \frac{dC_1(y_1(p_1, R_2(p_1)))}{dy_1}\left[\frac{\partial y_1}{\partial p_1} + \frac{\partial y_1}{\partial p_2}\frac{dR_2(p_1)}{dp_1}\right] = 0 \end{aligned} \qquad (29)$$

and this yields:

$$\begin{aligned} & y_1 \times \left[1 + \varepsilon + \eta \times \phi\right] = c_1 \times \frac{y_1}{p_1}\left[\varepsilon + \eta \times \phi\right] \Rightarrow \\ & p_1 = c_1 \frac{\varepsilon + \eta \times \phi}{1 + \varepsilon + \eta \times \phi} \end{aligned} \qquad (30)$$

where: $\eta = \frac{\partial y_1}{\partial p_2}\frac{p_2}{y_1}$ is the cross-price elasticity of demand and

$\phi = \frac{dR_2(p_1)}{dp_1}\frac{p_1}{R_2(p_1)}$ is the elasticity of the reaction of firm 2 to changes in firm 1's price.

D. Wages and Employment

The game is dynamic game in which:

1. The union chooses a wage rate, *w*
2. The firm observes the wage rate and chooses employment, *N*
3. The payoffs are *U(w,N)* for the union and π*(w,N)* for the firm

The firm chooses *N*, given *w*, so as to maximise:

$$\pi(w,N) = p \times f(N) - w \times N \quad (31)$$

The first-order conditions for this are:

$$w = p \times f'(N) \quad (32)$$

or wage equals the value of marginal product. Solving the first-order condition yields the firm's reaction function *N=N(w; p)*. The union, knowing this reaction function, chooses *w* so as to maximise:

$$U(w,N) = U(w,N(w)) \quad (33)$$

and the first order conditions for solving this are:

$$\frac{dU}{dw} + \frac{dU}{dN}\frac{dN}{dw} = 0 \Rightarrow -\frac{dU}{dw}/\frac{dU}{dN} = \left(\frac{dN}{dw}\right)_{U=\bar{U}} = \frac{dN}{dw} \quad (34)$$

Equation (34) can be written as:

$$\varepsilon_{Uw} = \frac{dU}{dw}\frac{w}{U} = -\left(\frac{dU}{dN}\frac{N}{U}\right)\left(\frac{dN}{dw}\frac{w}{N}\right) = \varepsilon_{UN} \times \eta \quad (35)$$

where: ε_{Uw} is the elasticity of utility (*U*) with respect to the wage rate (*w*), ε_{Uw} is the elasticity of utility with respect to employment (*N*) and η is the elasticity of labour demand with respect to the wage rate.

So, in equilibrium, the percentage gain in utility, consequent upon a percentage rise in wages, is equal to the percentage loss in utility from the loss in employment induced by wage increase.

2.2 Dynamic Games with Complete but Imperfect Information

In a two-stage game with complete information:

1. Players 1 and 2 *simultaneously* choose actions a_1 and a_2 from their action sets, A_1 and A_2, respectively.
2. Players 3 and 4 observe these choices a_1 and a_2, made from the first stage of the game, and choose *simultaneously* choose actions a_3 and a_4 from their action sets, A_3 and A_4, respectively.
3. The payoffs to each player are $u_i(a_1, a_2, a_3, a_4), i = 1...4$

To solve this game, suppose that for all feasible outcomes, $a_1 \in A_1, a_2 \in A_2$ there exists a Nash equilibrium for the second stage game between players 3 and 4: $a_3^*(a_1,a_2)$ and $a_4^*(a_1,a_2)$.
Players 1 and 2 know the Nash solutions for stage 2 of the game and, consequently, their game amounts to: choose a1 and a2 such that the payoffs are $u_i(a_1,a_2,a_3^*(a_1,a_2),a_4^*(a_1,a_2)), i=1,2$.
Suppose a_1^* and a_2^* represent the unique Nash equilibrium for this game. Then $(a_1^*,a_2^*,a_3^*(a_1^*,a_2^*),a_4^*(a_1^*,a_2^*)$ is the **subgame-perfect outcome** of the two-stage game.

A. Tariffs
First, the governments of two countries, i=1,2, choose tariffs t_1 and t_2, respectively. Then, the firm in country i observes the tariffs t_1 and t_2 and chooses quantities for home consumption and export: h_i and e_i , i=1,2.
The payoffs are:
1. the profits to firm 1 (in country 1) and 2 (in country 2) represented as:

$$\pi_i(t_i,t_j,h_i,e_i,h_j,e_j) = P(Y_i)h_i + P(Y_j)e_i - c\times(h_i+e_i) - t_j e_i,\ i,j=1,2,\ i\neq j \quad (36)$$

where: $Y_i = h_i + e_j$ is the total quantity sold in country i, and $t_i e_j$ is the tariff payment to country i, $i,j=1,2,\ i\neq j$.

2. The total welfare to country i where this is the sum of: Consumers' surplus; Firm profits; and Government revenue from the tariff:

$$W_i(t_1,t_2,h_1,e_1,h_2,e_2) =$$
$$\left[\int_0^{Y_i} P(Y_i)dY_i - P(Y_i)Y_i\right] + \pi_i(t_1,t_2,h_1,e_1,h_2,e_2) + t_i e_j,\ i,j=1,2, i\neq j \quad (37)$$

Suppose the countries have chosen their tariff rates t_1 and t_2 and suppose $h_i^*(t_i,t_j)$ and $e_i^*(t_i,t_j),\ i,j=1,2,i\neq j$ represent the Nash-equilibrium at the second stage of the game. Then by, definition, $h_i^*(t_i,t_j)$ and $e_i^*(t_i,t_j)$ solve:

$$\underset{h_i,e_i}{Max}\ \pi_i(t_i,t_j,h_i,e_i,h_j^*,e_j^*),\ i,j=1,2,i\neq j \quad (38)$$

Since the profit function in equation (36) can be separated into profits generated from home and from foreign sales, $h_i^*(t_i,t_j)$ and $e_i^*(t_i,t_j)$, respectively, solve:

$$\underset{h_i}{Max}\ \pi_i^i = P(Y_i)h_i - c\times h_i \text{ and } \underset{e_i}{Max}\ \pi_i^j = P(Y_j)e_i - c\times e_i - t_j\times e_i \quad (39)$$

$i,j=1,2,i\neq j$.
The first order conditions for this are:

$$P(Y_i) + P'(Y_i)h_i - c = 0 \text{ and } P(Y_j) + P'(Y_j)e_i - c - t_j \quad (40)$$

Solving the pair of equations in (40), yields the optimal values: $h_i^*(t_i,t_j)$ and $e_i^*(t_i,t_j)\ \ i,j=1,2,i\neq j$

In the first stage the two governments choose tariff rates: t_1, t_2 with the payoff functions:

$$W_i(t_i, t_j, h_i^*, e_i^*, h_j^*, e_j^*) \; i, j = 1, 2, i \neq j \tag{41}$$

If t_i^*, t_j^* represent a Nash-equilibrium then t_i^* solves:

$$\underset{t_i}{Max} \; W_i(t_i, t_j^*, h_i^*, e_i^*, h_j^*, e_j^*) \; i, j = 1, 2, i \neq j \tag{42}$$

and the first order conditions for this are:

$$\frac{\partial}{\partial t_i}\left[W_i(t_i, t_j^*, h_i^*, e_i^*, h_j^*, e_j^*)\right] = 0, i, j = 1, 2, i \neq j \tag{43}$$

solving which yields the ‘optimal’ tariff rates: t_1^*, t_2^*