Conditional Probabilities and Bayes' Theorem

Conditional Probabilities

Let A and B be two events. If N is the total number of possible outcomes and N_A and N_B are the number of outcomes favourable to A and B, respectively, then the probabilities of A and B, denoted P[A] and P[B], respectively are:

$P[A] = N_A / N$ and $P[B] = N_B / N$. These are the *unconditional* probabilities of A and B occurring.

Then the probability of A occurring, *given that B has occurred*, is the conditional probability of A, denoted $P[A \mid B]$ and the probability of B occurring, *given that A has occurred* is the conditional probability of B, denoted $P[B \mid A]$.

Example: A dice is thrown: let A be the event that the outcome is a "2" and let B be the event that the outcome is an even number. Then P[A]=1/6 and P[B]=1/2 and P[A|B]=1/3.

If N_{AB} are the number of occurrences favourable to *both A and B*:

$$P[A \mid B] = \frac{N_{AB}}{N_B} = \frac{N_{AB}}{N}\frac{N}{N_B} = \frac{P[AB]}{P[B]}$$

$$\text{By identical argument: } P[B \mid A] = \frac{N_{AB}}{N_A} = \frac{P[AB]}{P[A]} \qquad (1)$$

$$\Rightarrow P[A \mid B] = \frac{P[B \mid A]P[A]}{P[B]}$$

Given any two events A and B, we can write one of the events, say B as:

$$\begin{aligned} &B = (B \cap A) \cup (B \cap \overline{A}) \Rightarrow P[B] = P[AB] + P[\overline{A}B] \\ &\Rightarrow P[B] = P[B \mid A]P[A] + P[B \mid \overline{A}]P[\overline{A}] \end{aligned} \qquad (2)$$

where: $\overline{A}$ is the event "not-A".

Bayes' Theorem

Reverend Thomas Bayes – an 18^{th} century Presbyterian minister – proved what is, arguably, the most important theorem in statistics (see "In Praise of Bayes", The *Economist*, 28 September 2000).

Let T denote "theory" and D denote "data". Then the probability of the theorem being true, *given that the data has been observed*, is:

$$P(T \mid D) = \frac{P(TD)}{P(D)} = \frac{P(D \mid T)P(T)}{P(D)} \qquad (3)$$

where: $P(D)=P(D|T)P(T)+P(D|\overline{T})P(\overline{T})$, $\overline{T}$ being the event that the theory is false.

Interpretation: $P(T)$ is the *prior* probability of the theory being true. Given the evidence of the data, this prior probability is updated to arrive at the *posterior* probability, $P(T|D)$. The quantity, $P(D|T)/P(D)$ is the *updating* factor.

Example: Suppose there is a test for cancer. Let B be the event that the person tested has cancer and let A be the event that the test is positive, i.e. says the person has cancer. Suppose $P[A|B]=0.95$ and $P[\overline{A}|\overline{B}]=0.95$. We are interested in $P[B|A]$, the probability that a person has cancer, given that the test is positive:

$$P[B|A]=\frac{P[A|B]P[B]}{P[A]}=\frac{0.95\times P[B]}{P[A|B]P[B]+P[A|\overline{B}]P[\overline{B}]}$$
$$=\frac{0.95\times P[B]}{0.95\times P[B]+(0.05)(1-P[B])}$$

If $P[B]=0.005$, $P[B|A]=0.087$!

Suppose we desire that $P[B|A]=0.95$. What should $P[A|B]=P[\overline{A}|\overline{B}]$ be to achieve this? Solve for X in:

$$0.95=\frac{X\times 0.005}{X\times 0.005+(1-X)\times 0.995}\Rightarrow X=0.99974$$

Application to Asymmetric Information

The seller prices his cars, some at \$2500 and some at \$1000. He will always sell a good car for \$2500. He prices some of the bad cars at \$2500 and some at \$1000: the probability of a bad car being priced at \$2500 is μ and of it being priced at \$1000 is 1-μ; half of his cars are bad cars.

The buyer will accept a car priced at \$2500 with probability q and reject such a car with probability 1-q; the buyer will always buy a car priced at \$1000. The buyer believes that any car priced at \$2500 is a bad car with probability β and a good car with probability 1-β.

What is the probability that a bad car is sold for \$2500?

For the buyer:

$$\beta=P(B|p=2500)=\frac{P(p=2500|B)P(B)}{P(p=2500)}=\frac{0.5\mu}{0.5\mu+0.5} \quad (4)$$

Note: $P(p=2500)=P(p=2500\,|\,G)P(G)+P(p=2500\,|\,B)p(B)=0.5+0.5\mu$

If the buyer rejects the car priced at \$2500, his payoff is zero; if he accepts the \$2500 car then his payoff is:

$(3200-2500-1700)\beta+(3200-2500-200)(1-\beta)=-1000\beta+500(1-\beta)$. In equilibrium, his expected payoff from rejection or acceptance of a \$2500 car must be the same implying: $500(1-\beta)-1000\beta=0\Rightarrow\beta=\frac{1}{3}$.

Using this value of β to solve for μ: $\frac{1}{3}=\frac{0.5\mu}{0.5+0.5\mu}\Rightarrow\mu=\frac{1}{2}$

For the seller: If he offers a car for \$1000, his payoff is \$1000. If he offers a car for \$2500, his payoff is: $2500q+0(1-q)$. In equilibrium, the two payoffs are the same: $1000=2500q\Rightarrow q=\frac{2}{5}$

What is the probability that a bad car is sold for \$2500?

$$
\begin{aligned}
&P(\text{sold at \$2500}\,|\,B)=P(\text{offered at \$2500}\cap\text{accepted at \$2500|B})\\
&=\frac{P(\text{offered at \$2500}\cap\text{accepted at \$2500}\cap\text{B})}{P(B)}\\
&=\frac{P(\text{offered at \$2500}\cap B)P(\text{accepted at \$2500})}{P(B)} \qquad (5)\\
&=\frac{(1/2)\mu q}{(1/2)}=\frac{1}{2}\times\frac{2}{5}=0.2
\end{aligned}
$$

So seller can shift 20% of his stock of bad cars at the higher price of \$2500.